Locality-Sensitive Hashing

Algorithmic Problems Around the Web #5

Yury Lifshits
http://yury.name

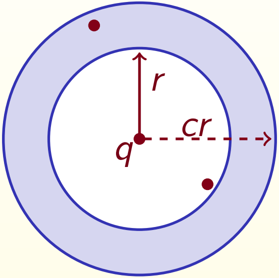
CalTech, Fall'07, CS101.2, http://yury.name/algoweb.html

Outline

Outline

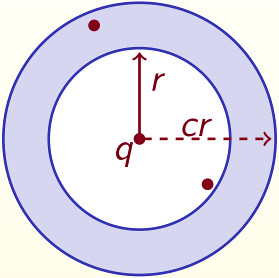
Approximate Algorithms

c-Approximate r-range query: if there at least one $p \in S: \; d(q,p) \le r$ return some $p': \; d(q,p') \le cr$



Approximate Algorithms

c-Approximate r-range query: if there at least one $p \in S: \; d(q,p) \le r$ return some $p' : \; d(q,p') \le cr$



c-Approximate nearest neighbor query: return some $p' \in S: \; d(p',q) \le cr_{NN}$, where $r_{NN} = \min_{p \in S} d(p,q)$

Today we consider only range queries

Today's Focus

Data models:

- d-dimensional Euclidean space: $\mathbb{R}^d$
- Hamming cube: $\{0,1\}^d$ with Hamming distance

Today's Focus

Data models:

- d-dimensional Euclidean space: $\mathbb{R}^d$
- Hamming cube: $\{0, 1\}^d$ with Hamming distance

Our goal: provable performance bounds

- Sublinear search time, near-linear preprocessing space
- Logarithmic search time, polynomial preprocessing space

Today's Focus

Data models:

- d-dimensional Euclidean space: $\mathbb{R}^d$
- Hamming cube: $\{0,1\}^d$ with Hamming distance

Our goal: provable performance bounds

- Sublinear search time, near-linear preprocessing space
- Logarithmic search time, polynomial preprocessing space

Still an open problem: approximate nearest neighbor search with logarithmic search and linear preprocessing

Part I

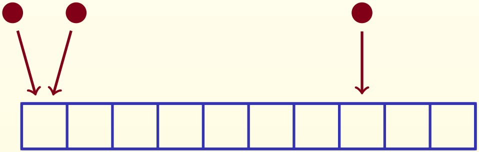
Locality-Sensitive Hashing: General Scheme

Definition of LSH

Indyk&Motwani'98

Locality-sensitive hash family $\mathcal{H}$ with parameters (c, r, P_1, P_2):

- If $\|p - q\| \leq r$ then $\mathcal{P}r_{\mathcal{H}}[h(p) = h(q)] \geq P_1$
- If $\|p - q\| \geq cr$ then $\mathcal{P}r_{\mathcal{H}}[h(p) = h(q)] \leq P_2$



The Power of LSH

Notation: $\rho = \frac{\log(1/P_1)}{\log(1/P_2)} < 1$

Theorem

Any (c, r, P_1, P_2)-locality-sensitive hashing leads to an algorithm for c-approximate r-range search with (roughly) n^ρ query time and $n^{1+\rho}$ preprocessing space

The Power of LSH

Notation: $\rho = \frac{\log(1/P_1)}{\log(1/P_2)} < 1$

Theorem

Any (c, r, P_1, P_2)-locality-sensitive hashing leads to an algorithm for c-approximate r-range search with (roughly) n^ρ query time and $n^{1+\rho}$ preprocessing space

Proof in the next four slides

LSH: Preprocessing

Composite hash function: $g(p) =< h_1(p), \ldots, h_k(p) >$

LSH: Preprocessing

Composite hash function: $g(p) =< h_1(p), \ldots, h_k(p) >$

Preprocessing with parameters L, k:

1. Choose at random L composite hash functions of k components each
2. Hash every $p \in S$ into buckets $g_1(p), \ldots, g_L(p)$

Preprocessing space: $\mathcal{O}(Ln)$

LSH: Search

1. Compute $g_1(q), \ldots, g_L(q)$
2. Go to corresponding buckets and explicitly check $d(p, q) \leq ?cr$ for every point there
3. Stopping conditions: (1) we found a satisfying object or (2) we tried at least $3L$ objects

Search time is $\mathcal{O}(L)$

LSH: Analysis (1/2)

In order to have probability of error at most δ we set k, L such that

$$P_2^k n \approx 1 \qquad L \approx (1/P_1)^k \log(1/\delta)$$

LSH: Analysis (1/2)

In order to have probability of error at most δ we set k, L such that

$$P_2^k n \approx 1 \qquad\qquad L \approx (1/P_1)^k \log(1/\delta)$$

Solving these constraints:

$$k = \frac{\log n}{\log(1/P_2)}$$

LSH: Analysis (1/2)

In order to have probability of error at most δ we set k, L such that

$$P_2^k n \approx 1 \qquad L \approx (1/P_1)^k \log(1/\delta)$$

Solving these constraints:

$$k = \frac{\log n}{\log(1/P_2)}$$

$$L = (1/P_1)^{\frac{\log n}{\log(1/P_2)}} \log(1/\delta) = n^{\frac{\log(1/P_1)}{\log(1/P_2)}} \log(1/\delta) = n^\rho \log(1/\delta)$$

LSH: Analysis (2/2)

The expected number of cr-far objects to be tried is $P_2^k Ln \approx L$

For true r-neighbor the chance to be hashed to the same bucket as q is at least
$1 - (1 - (1/P_1)^k)^L \geq 1 - (1/e)^{\frac{L}{(1/P_1)^k}} \geq 1 - \delta$

LSH: Analysis (2/2)

The expected number of cr-far objects to be tried is $P_2^k Ln \approx L$

For true r-neighbor the chance to be hashed to the same bucket as q is at least
$1-(1-(1/P_1)^k)^L \geq 1-(1/e)^{\frac{L}{(1/P_1)^k}} \geq 1-\delta$

Preprocessing space $\mathcal{O}(Ln) \approx n^{1+\rho+o(1)}$
Search $\mathcal{O}(L) \approx n^{\rho+o(1)}$

Part II

Andoni&Indyk'06 Hashing

Ball Grids Hashing: Idea

1. Apply low distortion embedding A into t-dimensional Euclidean space

Ball Grids Hashing: Idea

1. Apply low distortion embedding A into t-dimensional Euclidean space
2. Set up U $4w$-step grids of w-radius balls that all together cover t-dimensional space

Ball Grids Hashing: Idea

1. Apply low distortion embedding A into t-dimensional Euclidean space
2. Set up U $4w$-step grids of w-radius balls that all together cover t-dimensional space
3. Hash object p to the id of the first ball covering $A(p)$

BG Hashing: Initialization

Parameters: $t = \log^{2/3} n, w = r \log^{1/6} n, U = 2^{t \log t} \log n$

- Construct $d \times t$ matrix A taking every element at random from normal distribution $N(0, \frac{1}{\sqrt{t}})$
- For every $1 \leq i \leq U$ choose a random shift $\bar{v}_i \in [0, 4w]^t$

BG Hashing: Computing

1. Compute $p' = A(p)$
2. From $i = 1$ to U check whether p' is covered by i-th grid of balls. If so return i and ball's center and stop.
3. If no such ball found return FAIL

BG Hashing: Analysis

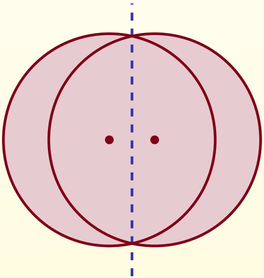
Fact: Probability of $\frac{\|Ap - Ap'\|}{\|p - p'\|} \notin [1 - \varepsilon, 1 + \varepsilon]$ is at most $\exp(-\varepsilon^2 t)$

BG Hashing: Analysis

Fact: Probability of $\frac{\|Ap-Ap'\|}{\|p-p'\|} \notin [1-\varepsilon, 1+\varepsilon]$ is at most $\exp(-\varepsilon^2 t)$

Given two points $p, s \in \mathbb{R}^t : \|p - s\| = \Delta$:

$$Pr[h(p) = h(s)] = \frac{B(p,w) \cap B(s,w)}{B(p,w) \cup B(s,w)}$$



BG Hashing: Final Result

3-pages computational proof:

$$\rho = \frac{\log(1/P_1)}{\log(1/P_2)} = 1/c^2 + o(1)$$

BG Hashing: Final Result

3-pages computational proof:

$$\rho = \frac{\log(1/P_1)}{\log(1/P_2)} = 1/c^2 + o(1)$$

Theorem (Andoni & Indyk 2006)

Consider c-approximate r-range search in d-dimensional space. Then for every δ there is a randomized algorithm with (roughly) $n^{1/c^2+o(1)}$ query time and $n^{1+1/c^2+o(1)}$ preprocessing space. For every query this algorithm answers correctly with probability at least $1-\delta$

Future of LSH

Achievements:

- Provably sublinear search time
- Utilization of low-distortion embedding

Future of LSH

Achievements:

- Provably sublinear search time
- Utilization of low-distortion embedding

Current drawbacks:

- Probability of error can not be amplified only in preprocessing stage, it can not be decreased to $1/n$
- Asymptotic analysis of power degree: from what place $n^{1/c^2+o(1)}$ is really sublinear?
- For nearest neighbor search $c = \max \frac{r_{NN}(q)}{r_{FN}(q)}$, where $r_{FN}(q)$ is the farthest neighbor. This might be pretty close to 1

Exercise

Prove that $2^{\mathcal{O}(t)}$ number of randomly chosen $(w, 4w)$ ball grids is enough to cover t-dimensional space with probability $1/2$

Exercise

Prove that $2^{\mathcal{O}(t)}$ number of randomly chosen $(w, 4w)$ ball grids is enough to cover t-dimensional space with probability $1/2$

Thanks for your attention! Questions?

References

Course homepage http://yury.name/algoweb.html

Y. Lifshits
The Homepage of Nearest Neighbors and Similarity Search
http://simsearch.yury.name

A. Andoni, P. Indyk
Near-Optimal Hashing Algorithms for Approximate Nearest Neighbor in High Dimensions. FOCS'06
http://web.mit.edu/andoni/www/papers/cSquared.pdf